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SECONDARY CONJUGATE SURFACES

BY T. SMITH, M.A., F.INST.P.
(Optics Department, The National Physical Laboratory.)

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ABSTRACT. Examples are given of conicoids which satisfy the secondary object-image relation for refraction at a spherical surface with a small stop. The special cases where the conicoid degenerates into two planes are considered. It is shown that if the object conicoid is of this type the image conicoid is only similarly degenerate in a trivial case, and the image of a plane is therefore a portion of a conicoid. Refraction at the vertex of a single thin lens leads to relations of the same general character. The inclusion of simple results of this type in courses on geometrical optics is suggested as a corrective to the usual assumption that the image of a plane is itself a plane.

General relations between the curvatures of object and image curves lying in any axial plane formed by axial refraction at a combination of thin lenses are given.

Introduction

IT is an almost universal custom when a pupil is being instructed in geometrical optics to deceive him (with the most benevolent intentions) into thinking that in real optical instruments a plane surface ordinarily has a plane image. If the teacher is abnormally conscientious he may point out that this assumption is not quite strictly true. The pupil is then left happy in the belief that aberrations are defects of comparatively small magnitude, that such niceties are only of significance to the technical designer of instruments, and that he himself need neither encounter them nor be troubled by them in experimental work. Any suspicions the over-alert or the unusually obstinate pupil may express regarding the stability of the foundations on which the teaching is to rest will doubtless be lulled to sleep by iteration of this theme. I imagine it was because he knew how far this doctrine departs from the truth that the late Prof. Silvanus Thompson denounced so emphatically the English optical treatises of his day. The difficulties of the technical optics of the standard German treatises, which he apparently would have taken as the basis for teaching instead, are too great for this alternative course to be adopted. Any appreciable

numerical work, such as ray tracing, apart from other objections, is open to the insuperable objection that adequate time cannot be spared for it. Teachers will probably require a substitute which merits the title of geometrical optics at least as fairly as the matter it displaces. The only direction in which there appears to be much hope of finding an acceptable solution of the very real difficulties this condition offers leads to the elucidation of the relations between simple, but not in general plane, conjugate surfaces. The present paper will be amply justified if it leads teachers to consider whether such a solution can be adopted. The method of presentation has been dictated by the need for brevity. Other methods of greater interest to the average pupil, covering the range of this paper, will readily be constructed by the teacher. The conclusions may prove of value to the lens designer in considering general plans for correcting the grosser defects of instruments.

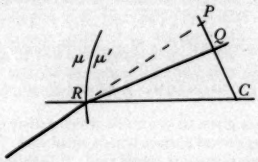

Fig. 1.

As far as is known the results to be given are in general new, but references have been made on previous occasions to examples of spherical secondary surfaces. It may not be superfluous to remark that the so-called secondary surfaces and foci are of greater importance than the primary surfaces and foci, which are found by plane ray tracing. E. W. Taylor*, experimenting with an achromatic telescope objective corrected for spherical aberration and coma, was able to locate the primary and secondary foci accurately over almost the whole of the field for which light was transmitted by the lens. He found that the secondary principal foci lay to a high order of accuracy on a sphere. It is well known that for an ideally thin single lens with a small stop at the common vertex of the two surfaces this surface is strictly spherical, and I have shown† that for any combination of thin lenses in contact the departure from a sphere is small. Several examples where both conjugate surfaces are spherical have been given in discussing the properties of a single spherical surface‡. I now propose to consider the relations between such conjugate surfaces more generally.

* *Trans. Opt. Soc.* **24** 1922–3 (321).
† *Trans. Opt. Soc.* **25** 1923–4 (89).
‡ *Trans. Opt. Soc.* **28** 1926–7 (225).

The general theory

Referring to Fig. 1, let RP and RQ be incident and refracted rays which at R meet the spherical refracting surface $\mathscr{S}$ whose centre is C. The points P and Q are secondary conjugate foci if P, Q and C are collinear. Since the sines of the angles PRC, QRC are inversely proportional to μ, μ', the two refractive indices, we have from two expressions for the ratio of the areas of the triangles PRC, QRC,

$$\frac{QC}{PC} = \frac{RQ}{\mu'} \cdot \frac{\mu}{RP}.$$

Thus if C is taken as the origin of coordinates, and (x, y, z) and (x', y', z') represent the points P and Q respectively, the relations* connecting conjugate points are

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} = \frac{w'}{w} = \frac{q}{\mu'} \cdot \frac{\mu}{p},$$

where p and q represent RP and RQ respectively, and w and w' have for convenience been added to denote $(x^2 + y^2 + z^2)^{\frac{1}{2}}$ and $(x'^2 + y'^2 + z'^2)^{\frac{1}{2}}$, that is they represent the lengths PC and QC. It follows from these relations that if the equation of an object surface $\mathscr{C}$ can be thrown into the form

$$\phi\,(x, y, z, p/\mu) = 0,$$

that of the corresponding image surface $\mathscr{C}'$ will be

$$\phi\,(x', y', z', q/\mu') = 0,$$

where ϕ is a homogeneous function of all its variables. The special conditions of each problem are contained in the particular forms assumed by p and q.

Linear relations

It will probably be conceded that the value of any relations we can establish will be in the inverse order of the equation $\phi = 0$, when the function ϕ is, if necessary, transformed into a rational integral polynomial in the three coordinates. We therefore consider in the first place the equations of this group which are linear in x, y, z. The most general equation permissible is evidently

$$lx + my + nz = 0,$$

and as this represents only planes through the centre C of the refracting sphere $\mathscr{S}$ it is trivial from the present point of view. (The same remark applies to all equations not involving p and q. They merely represent the self-conjugate cones of rays which have C as their vertex.) We thus reach the following important theorem.

* We may note that the usual equation of refraction

$$\frac{\mu'}{q} - \frac{\mu}{p} = \frac{\mu' \cos\phi' - \mu \cos\phi}{r}$$

is a mere rearrangement of the obvious relation

$$\frac{r - q\cos\phi'}{w'} = \frac{r - p\cos\phi}{w}.$$

THEOREM I. *When refraction takes place at a spherical surface $\mathscr{S}$, two plane surfaces cannot be conjugate to one another except in the trivial self-conjugate case.*

Our next step is to consider surfaces of the second order. It will be convenient to discuss a number of separate cases determined by the position of the effective stop.

Central stop

Of the various positions the stop can take special importance can be claimed for the centre of curvature C of the spherical surface $\mathscr{S}$. The images formed by rays which pass close to this point are stigmatic. The results obtained in this section thus apply to the primary as well as to the secondary conjugate surfaces.

The symmetry of the system around C enables us to infer the next result without other investigation.

THEOREM II. *When the stop is at the centre of curvature of the refracting surface, the surfaces of constant magnification are spheres with the same centre. In particular the images of infinitely distant objects lie on a sphere $\mathscr{S}'$.*

To deduce this formally from the general equations, we note that $p = r + w$, $q = r + w'$, where r is the radius of $\mathscr{S}$. Since the surfaces $\frac{p}{\mu} + \alpha w = 0$ and $\frac{q}{\mu'} + \alpha w' = 0$ are conjugate, we have, on substituting for p and q,

$$w = -\frac{r}{1 + \alpha\mu}, \quad w' = -\frac{r}{1 + \alpha\mu'},$$

and the magnification m is given by $m = w'/w$, or $m = \frac{1 + \alpha\mu}{1 + \alpha\mu'}$, i.e. the constant magnification surfaces are

$$w' = mw = -\frac{r(m\mu' - \mu)}{\mu' - \mu} = -\frac{(m\mu' - \mu)}{\kappa},$$

where κ is the refracting power of $\mathscr{S}$. This is the form required by the theorem just stated. It should be observed that we obtain the same rationalised equation whichever sign is given to w and w', but in particular cases we must be prepared to pay special attention to the signs these quantities bear.

The equations we have just formed are clearly less general than they need be to yield rationalised equations of the second order. More general forms for conjugate surfaces are

$$\frac{p}{\mu} + \alpha w = \beta x + \gamma y + \delta z,$$

$$\frac{q}{\mu'} + \alpha w' = \beta x' + \gamma y' + \delta z'.$$

Without loss of generality we may put $\gamma = \delta = 0$. Now the equation

$$\pm w = e(x - x_0)$$

represents the surface obtained by rotating a conic about its major axis. The origin is

a focus of the curve, e is the eccentricity, and x_0 is the distance of the focus from the directrix. Comparing the equations we have, with an obvious use of accents,

$$\mu\beta = e\,(1 + \mu\alpha), \quad \mu'\beta = e'\,(1 + \mu'\alpha), \quad \mu x_0\beta = \mu' x_0'\beta = r.$$

The radius of curvature of a conic at its vertex is equal to the semi-latus-rectum, i.e. it is equal to ex_0. We may express these results in the following form:

THEOREM III. *When the stop is at the centre of curvature C of the refracting surface $\mathscr{S}$, if $\mathscr{C}$ is a conicoid of revolution (circular paraboloid, prolate spheroid, hyperboloid of two sheets) of which C is a focus, its conjugate $\mathscr{C}'$ is also a conicoid of revolution having C as focus and possessing rotational symmetry about the same axis. The distances of the directrix planes from C when expressed in wave-lengths are equal ($\mu x_0 = \mu' x_0'$). The eccentricities e, e' are connected with the refracting power κ of $\mathscr{S}$ by the relation*

$$\frac{1}{e} - \frac{1}{e'} = \frac{\kappa x_0}{\mu'} = \frac{\kappa x_0'}{\mu}.$$

This is essentially the same relation as

$$\frac{T}{\mu} - \frac{T'}{\mu'} = \frac{\kappa}{\mu\mu'},$$

where T and T' are the curvatures of the generating conics of $\mathscr{C}$ and $\mathscr{C}'$ at their vertices.

The quantity $\kappa/\mu\mu'$ will be recognised as the single-surface equivalent of the Airy-Petzval sum. It should be noted that no general statement can be made about the direct curvature difference $T - T'$, since this involves the parameter α used in specifying particular surfaces $\mathscr{C}$, $\mathscr{C}'$.

Since a pair of planes is a special form of a conicoid, we should expect to obtain the conjugate of a plane by selecting suitable conicoids in the object space*. We readily see that the necessary condition is $1 + \alpha\mu = 0$. This gives $e = \infty$ and $T = 0$, and the conicoid coincides with the directrix plane. We thus obtain:

THEOREM IV. *When the stop is at the centre of curvature C of the refracting surface $\mathscr{S}$, the surface $\mathscr{C}'$ conjugate to a plane $\mathscr{P}$ is a conicoid of revolution having C as focus and the normal to $\mathscr{P}$ as its axis of symmetry. The semi-latus-rectum of $\mathscr{C}'$ is equal to the optical focal length of $\mathscr{S}$, and the distance from C to the directrix plane of $\mathscr{C}'$ is equal to the distance of $\mathscr{P}$ from C provided both distances are expressed in wave-lengths.*

The common value of the semi-latus-rectum is clearly an immediate consequence of rotational symmetry. It will also be clear that the complete surface $\mathscr{C}'$ represents the image formed by all parts of the refracting surface $\mathscr{S}$. In general $\mathscr{C}'$ is divided into two parts by the plane through C parallel to $\mathscr{P}$, one of which is the locus of the image formed by the portion of the refracting surface which is concave to $\mathscr{P}$, the other by the remainder which is convex to $\mathscr{P}$.

It is of interest to consider how $\mathscr{C}'$ changes as the plane $\mathscr{P}$ is moved parallel to itself. If $\mathscr{P}$ is the principal focal plane of present day geometrical optics, its image

* Since this paper was written a remark by P. Culmann appearing on p. 135 of M. v. Rohr's *The Formation of Images in Optical Instruments* has been pointed out to me. Speaking of central refraction at a spherical surface he says "It can easily be demonstrated that a plane has corresponding to it a surface of revolution of the second order."

is a paraboloid. As $\mathscr{P}$ recedes from the surface $\mathscr{S}$, e diminishes, and the surfaces $\mathscr{C}'$ are ellipsoids. When $\mathscr{P}$ reaches infinity $\mathscr{C}'$ is spherical.

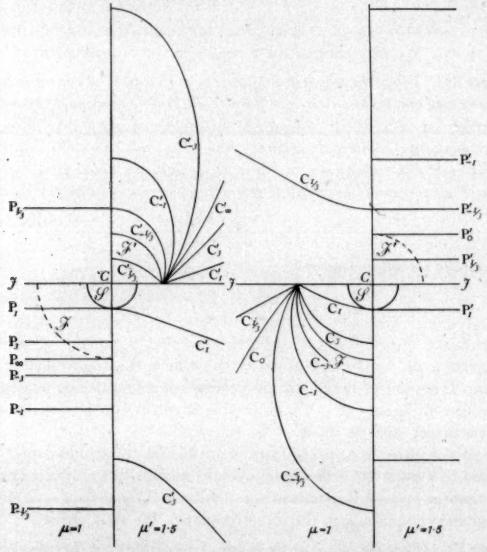

Fig. 2. Fig. 3.

Figs. 2 and 3. Plane surfaces and their images formed by light which passes through a small aperture at C, the centre of curvature of a spherical refracting surface $\mathscr{S}$.

In the left half of each figure the surfaces in the medium of refractive index $\mu = 1$, which lies below $\mathscr{S}$, are shown. In the right half the conjugate surfaces in the medium of refractive index $\mu' = 1{\cdot}5$, which extends above $\mathscr{S}$, are shown. In Fig. 2 the former, and in Fig. 3 the latter, are planes. $\mathscr{F}$ and $\mathscr{F}'$ are the spheres conjugate to planes at infinity. The numerical suffixes give the magnification of the constant magnification surfaces which the planes and conicoids touch on the axis. They are the magnifications associated with the planes in collinear optics. $\mathscr{J}C\mathscr{J}$ is the self-conjugate plane.

For clearness the diagrams are drawn for refraction in the lower half of the sphere $\mathscr{S}$ only.

If we return to the principal focal plane and move towards C, the surfaces $\mathscr{C}'$ are hyperboloids of two sheets. Parallels through C to the asymptotic cones clearly meet $\mathscr{P}$ in the circle in which $\mathscr{P}$ and the object space principal focal sphere $\mathscr{F}$ intersect. The eccentricity increases as the distance of $\mathscr{P}$ from C diminishes, and both sheets of the hyperboloid become flattened out and coincide with $\mathscr{P}$ when it passes

through C. As the properties of the system are symmetrical about C this description of events is exhaustive. If it is realised that the sphere conjugate to infinity separates the images of real from those of virtual objects, the correlation of the image point with the point of refraction becomes very simple.

Fig. 2 illustrates the relations between $\mathscr{P}$ and $\mathscr{C}'$ when $\mu'/\mu = 1{\cdot}5$, and Fig. 3 those between $\mathscr{P}'$ and $\mathscr{C}$ for the same indices.

The various results described here enable us to form a complete picture of tne conjugate relations for central refraction at a spherical surface.

Surface stop

Central refraction is chiefly significant because it is stigmatic. From the nature of the case it is not typical of refraction in general, and we must therefore supplement the information we have obtained by placing a stop elsewhere. We should expect to find simple results when the stop is centered at a point on the surface of the refracting sphere, for this point will then be common to every ray traversing the system. If we take (i, j, k) as the coordinates of the stop centre where $i^2 + j^2 + k^2 = r^2$, we have

$$p^2 = (x-i)^2 + (y-j)^2 + (z-k)^2, \quad q^2 = (x'-i)^2 + (y'-j)^2 + (z'-k)^2,$$

and the general equations of conjugate second order surfaces* are

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + \frac{1}{\mu^2}\{(x-i)^2 + (y-j)^2 + (z-k)^2\} = 0$$

and

$$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' + \frac{1}{\mu'^2}\{(x'-i)^2 + (y'-j)^2 + (z'-k)^2\} = 0$$

respectively. This gives by inspection:

THEOREM V. *If the centre of the stop is placed where the plane $\mathscr{A}$ touches the refracting sphere $\mathscr{S}$ of which C is the centre, the conicoid $\mathscr{C}$ will have another conicoid $\mathscr{C}'$ for its image if $\mathscr{A}$ is the polar plane of C with respect to $\mathscr{C}$. $\mathscr{A}$ will also be the polar plane of C with respect to $\mathscr{C}'$, and the polar planes of the stop centre with respect to $\mathscr{C}$ and $\mathscr{C}'$ are coincident. The principal axes of $\mathscr{C}$ and $\mathscr{C}'$ are parallel to one another. If one is a surface of revolution so is the other, and if one is a sphere so also is the other.*

The equations given above for $\mathscr{C}$ and $\mathscr{C}'$ are very convenient for practical application, but it is of interest to see the metrical relations between the two conicoids. For this purpose we take the axes of coordinates parallel to the axes of the conicoids, and introduce variables to represent the squares of their semi-axes. Let then

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} + \frac{1}{\mu^2}\{(x-i)^2 + (y-j)^2 + (z-k)^2\} = 0$$

and

$$\frac{(x - ia/S)^2}{a} + \frac{(y - j\beta/S)^2}{\beta} + \frac{(z - k\gamma/S)^2}{\gamma} = 1$$

* This statement refers to the principal group of second order surfaces: this group is not however exhaustive of those which occur with a surface stop.

be two forms of the equation for $\mathscr{C}$. The relations between the two sets o fconstants are

$$\alpha(1+\mu^2/a)=\beta(1+\mu^2/b)=\gamma(1+\mu^2/c)=S=\frac{r^2}{-1+(i^2\alpha+j^2\beta+k^2\gamma)/S^2}.$$

If we eliminate α, β, γ we obtain

$$S+r^2=\frac{i^2}{1+\mu^2/a}+\frac{j^2}{1+\mu^2/b}+\frac{k^2}{1+\mu^2/c},$$

or

$$S+\mu^2\left(\frac{i^2}{a+\mu^2}+\frac{j^2}{b+\mu^2}+\frac{k^2}{c+\mu^2}\right)=0.$$

Similarly

$$S'+\mu'^2\left(\frac{i^2}{a+\mu'^2}+\frac{j^2}{b+\mu'^2}+\frac{k^2}{c+\mu'^2}\right)=0,$$

and

$$\alpha'(1+\mu'^2/a)=\beta'(1+\mu'^2/b)=\gamma'(1+\mu'^2/c)=S'.$$

Thus if we know α, β, γ, S we can find a, b, c and thence S', α', β', γ'.

If we eliminate a, b, c between the equations for α, β, γ, α', β', γ' we obtain

$$\frac{S'}{\mu'^2\alpha'}-\frac{S}{\mu^2\alpha}=\frac{S'}{\mu'^2\beta'}-\frac{S}{\mu^2\beta}=\frac{S'}{\mu'^2\gamma'}-\frac{S}{\mu^2\gamma}=\frac{1}{\mu'^2}-\frac{1}{\mu^2}.$$

Multiplying the first three by l^2, m^2, n^2 respectively and adding, where l, m, n are the direction cosines of an arbitrary direction, we have

$$\frac{S'}{\mu'^2d'^2}-\frac{S}{\mu^2d^2}=\frac{1}{\mu'^2}-\frac{1}{\mu^2},$$

where d and d' are any parallel semi-diameters of $\mathscr{C}$ and $\mathscr{C}'$. It is clearly of importance to understand the meaning of S. Let s be the length of the perpendicular from the centre of $\mathscr{C}$ to $\mathscr{A}$, and suppose it has the same sign as r if the centres of $\mathscr{S}$ and $\mathscr{C}$ lie on the same side of $\mathscr{A}$. Then

$$s=r-(i^2\alpha+j^2\beta+k^2\gamma)/Sr,$$

or substituting in the original relations

$$S+sr=0.$$

The equation connecting d and d' may therefore be written

$$\frac{s'}{\mu'^2d'^2}-\frac{s}{\mu^2d^2}=\frac{1}{r}\left(\frac{1}{\mu^2}-\frac{1}{\mu'^2}\right).$$

Before leaving this equation a caution may be given against writing the right side of the last equation with the wrong sign. Since the equation means that the left side is independent of the conjugate surfaces considered, we may be tempted to suppose that we can put $s=s'=d=d'=r$, on the ground that the image of the refracting surface coincides with the surface itself. If however we take the aplanatic surfaces we have $s=s'=r$, $\mu'd'=\mu r$, $\mu d=\mu' r$, agreeing with the equation as written. The explanation of the fallacy is that the refracting surface is self-conjugate *except when the stop lies on its surface*. When the source of the mistake is pointed out it is easy to see that in the special circumstances with which we are dealing the image of the refracting

surface is far from coinciding with the surface. The result we have to add to the last theorem is:

THEOREM VI. *The ratio of the distance of the centre of $\mathscr{C}$ from $\mathscr{A}$ to the square of any diameter of $\mathscr{C}$ measured in wave-lengths differs from the corresponding ratio for $\mathscr{C}'$ when the two diameters are parallel, by a magnitude which is constant for the given refracting surface.*

The surfaces of constant magnification m are obtained by imposing the condition that the substitution

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} = m$$

transforms the equation for $\mathscr{C}$ into that for $\mathscr{C}'$, and vice versa. This yields a result which has been employed previously, viz.:

THEOREM VII. *When the stop lies on the refracting surface the surfaces of constant magnification m are spheres with their centres on the diameter through the centre of the stop. The radii are respectively ρ/m and ρ, and the distances of their centres from the stop are $\mu\rho/\mu' m$ and $\mu'\rho/\mu$, where $\rho = \dfrac{(1-m)\,\mu\mu' r}{\mu'^2 - \mu^2}$ (a positive value of the distance of a centre from the stop implies that this centre is on the same side of the stop as the centre of the refracting sphere). The centre of the stop is thus a centre of similitude for the constant magnification surfaces of both spaces.*

We now consider the case when $\mathscr{C}$ breaks down into two planes $\mathscr{P}_1, \mathscr{P}_2$. It is easy to see that it must be possible to express the part of the expression for $\mathscr{C}$ which does not involve p in the form

$$(ix + jy + kz - r^2)^2 - (Ix + Jy + Kz)^2 - r^2\,[(x-i)^2 + (y-j)^2 + (z-k)^2].$$

It follows that the equation of $\mathscr{C}$ is

$$[(i+I)\,x + (j+J)\,y + (k+K)\,z - r^2]\,[(i-I)\,x + (j-J)\,y + (k-K)\,z - r^2] = 0,$$

and that of $\mathscr{C}'$ is

$$\mu'^2\,[(Ix + Jy + Kz)^2 - (ix + jy + kz - r^2)^2] + (\mu'^2 - \mu^2)\,r^2\,[(x-i)^2 + (y-j)^2 + (z-k)^2] = 0.$$

From the form of $\mathscr{C}$ we obtain:

THEOREM VIII. *When the stop is at the point of contact of a plane $\mathscr{A}$ with the refracting sphere $\mathscr{S}$ of which C is the centre, the image of two planes $\mathscr{P}_1, \mathscr{P}_2$ will be a conicoid if the three planes $\mathscr{A}, \mathscr{P}_1, \mathscr{P}_2$ have a common line of intersection and if any straight line through C is divided harmonically by the point C and the planes $\mathscr{P}_1, \mathscr{A}, \mathscr{P}_2$.*

This implies that one plane can always be selected arbitrarily and independently of the position of the stop. The second plane is merely the image of the parts of the conicoid $\mathscr{C}'$ which are not conjugate to the first plane.

By Theorem V the principal axes of $\mathscr{C}'$ will be parallel to the line of intersection of $\mathscr{P}_1$ and $\mathscr{P}_2$ and to the internal and external bisectors of the angle which $\mathscr{P}_1$ and $\mathscr{P}_2$ make with one another. In fact we may put

$$\mathscr{C} = \mathscr{P}_1\mathscr{P}_2 = \left(ix + \epsilon jy - \frac{r^2\epsilon}{1+\epsilon}\right)\left(ix - \epsilon jy + \frac{r^2\epsilon}{1-\epsilon}\right) = 0,$$

which is a function of ϵ^2 only since a change in the sign of ϵ is equivalent to interchanging $\mathscr{P}_1$ and $\mathscr{P}_2$. The centre is the point $z = 0$ of the line

$$-ix = \epsilon^2 jy = \frac{r^2\epsilon^2}{1-\epsilon^2}.$$

The equation for $\mathscr{C}'$ is

$$A(Bx + \mu^2 i)^2 + B(Ay + \mu^2 j)^2 + (\mu'^2 - \mu^2)\,ABz^2 - (\mu'^2 - \mu^2)\,\mu^2\mu'^2 D^2 r^2\epsilon^2 = 0,$$

where $$A = \mu'^2 D\,\epsilon^2 - \mu^2, \quad B = \mu'^2 D - \mu^2, \quad D = \frac{i^2 + j^2\epsilon^2}{r^2\epsilon^2}.$$

If the magnification for the point $z = \zeta$ on the line of intersection of $\mathscr{P}_1$ and $\mathscr{P}_2$ is denoted by m, we have

$$(1-m)^2\mu^2 r^2 = m^2(\mu'^2 - \mu^2)\left[\zeta^2 + \frac{r^2(i^2 + j^2\epsilon^2)^2}{i^2 j^2 (1-\epsilon^2)^2}\right],$$

so that there are two values of m connected by the condition $\frac{1}{m_1} + \frac{1}{m_2} = 2$. Thus each point of this line is represented by two points in the image; and the object point, the two image points, and the centre C form a harmonic range. The explanation of this abnormality is that we have two distinct rays refracted at the aperture which pass through any object point in the plane $\mathscr{A}$, viz. that which has actually passed through the point and that which is proceeding in the opposite direction towards the point. These are refracted as opposite generators of the cone which limits the real field that can be imaged through this particular aperture. It is clear that we can only gain any information about objects in the neighbourhood of this object point from an examination of one of the image neighbourhoods, and it is easy to select the one which is of real significance.

It is worth observing that $\mathscr{C}'$ is never an imaginary surface, so that a second order image of any plane always exists for refraction at any point of the surface of the sphere.

If the two planes $\mathscr{P}_1$, $\mathscr{P}_2$ are parallel they are normal to the diameter of the sphere on which the stop lies, and $\mathscr{C}'$ is then a surface of revolution about this axis. It is an ellipsoid if $\mu'^2 - \mu^2\epsilon^2$ has the same sign as $\mu'^2 - \mu^2$, an hyperboloid if the signs are opposed, and a paraboloid when $\epsilon = \pm\,\mu'/\mu$, that is when the plane is tangent to the aplanatic sphere.

Aplanatic stops

The two positions of the stop we have already considered are the only possible examples of self-conjugate stops. In every other position the incident and emergent stops are separated from one another. Of these separated positions much the most important examples occur when the stop centres lie at conjugate points of the aplanatic spheres. Under these conditions the one centre is the image of the other for all rays traversing the refracting surface.

The conditions it is convenient to apply can be derived from the fundamental equations involving p and q, but it is simple to derive them directly. In Fig. 4 let

P and Q be conjugate points on rays refracted at R, a point on the surface of a sphere of which C is the centre, and suppose the incident and refracted rays have passed through two conjugate points A and B. Taking two expressions for the areas of the triangles PAC, QBC, we have

$$\frac{CA.CP}{CB.CQ} = \frac{RP - RA}{\mu} \cdot \frac{\mu'}{RQ - RB} = \frac{AP}{BQ} \cdot \frac{\mu'}{\mu}.$$

At the aplanatic surfaces $\mu^2 CA = \mu\mu' r = \mu'^2 CB$, and thus if p' and q' are the distances of a pair of conjugate points from the aplanatic surface, we have

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} = \frac{w'}{w} = \frac{\mu' q'}{\mu p'},$$

from which it follows that the surfaces

$$\mathscr{C} = \phi(x, y, z, w, \mu p') = 0$$

and

$$\mathscr{C}' = \phi(x', y', z', w', \mu' q') = 0$$

are conjugate if ϕ is homogeneous.

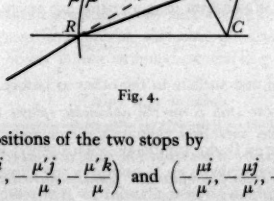

Fig. 4.

If we denote the positions of the two stops by

$$\left(-\frac{\mu' i}{\mu}, -\frac{\mu' j}{\mu}, -\frac{\mu' k}{\mu}\right) \text{ and } \left(-\frac{\mu i}{\mu'}, -\frac{\mu j}{\mu'}, -\frac{\mu k}{\mu'}\right),$$

where the signs are chosen to be consistent with real refraction at (i, j, k) on the refracting surface, the equations for $\mathscr{C}$ and $\mathscr{C}'$ take the forms*

$$\mathscr{C} = \mu'^2 r^2 + 2\mu\mu'(ix + jy + kz) + \mu^2 w^2 + ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

and

$$\mathscr{C}' = \mu^2 r^2 + 2\mu\mu'(ix' + jy' + kz') + \mu'^2 w'^2 + ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' = 0.$$

We can at once infer:

THEOREM IX. *When the stop lies in the appropriate aplanatic surface, the conditions which must be fulfilled for a conicoid $\mathscr{C}$ to have another conicoid $\mathscr{C}'$ as its secondary image are as follows:*

(i) *the polar planes of C, the centre of the refracting sphere with respect to $\mathscr{C}$ and $\mathscr{C}'$, are the tangent planes $\mathscr{A}$, $\mathscr{B}$, to the aplanatic surfaces through the stop and its image;*

* As with the surface stop there is another series of second order surfaces.

(ii) *the polar planes of the stop and its image with respect to the corresponding conicoids are coincident;*

(iii) *the principal axes of $\mathscr{C}$ and $\mathscr{C}'$ lie in parallel directions; if one is a surface of revolution so is the other, and if one is a sphere so also is the other;*

(iv) *if t and t' are the distances of the centres of $\mathscr{C}$ and $\mathscr{C}'$ from $\mathscr{A}$ and $\mathscr{B}$ respectively, and d and d' are the lengths of parallel semi-diameters of $\mathscr{C}$ and $\mathscr{C}'$, then*

$$\frac{t'}{d'^2} - \frac{t}{d^2} = \frac{\mu^2 - \mu'^2}{\mu\mu' r}$$

(positive signs are to be given to t and t' if the centres of $\mathscr{C}$ and $\mathscr{C}'$ lie respectively on the same sides of $\mathscr{A}$ and $\mathscr{B}$ as the centre C of the refracting sphere).

For reasons similar to those given in discussing the relations with a surface stop, we may take the refracting surface as a self-conjugate surface with

$$d = d' = r, \quad \mu^2 t = \mu'^2 t' = \mu\mu' r,$$

but we may not take the aplanatic surfaces as object and image, for this would give $d = t$, $d' = t'$ with the same values of t as before, so that

$$\frac{t'}{d'^2} - \frac{t}{d^2} = \frac{\mu'^2 - \mu^2}{\mu\mu' r},$$

i.e. the sign of the right side is reversed.

To find the surfaces of constant magnification we apply the conditions

$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z} = m$$

for the transformation of one surface to the other as before. We are thus led to

THEOREM X. *When the stop is on the aplanatic sphere the surfaces of constant magnification m are spheres with their centres on the diameter of the refracting sphere which passes through the centre of the stop. The radii are respectively ρ/m and ρ, and the distances of their centres from the aplanatic points which define the stop centres are $\mu'\rho/\mu m$ and $\mu\rho/\mu'$, where*

$$\rho = \frac{(m\mu'^2 - \mu^2) r}{\mu'^2 - \mu^2}$$

(a positive value of the distance of a centre from the stop implies that this is on the same side of the stop as the centre of the refracting sphere). Each stop centre is thus a centre of similitude for the associated system of constant magnification surfaces.

If $\mathscr{C}$ degenerates into two planes the equations of the two conicoids take the form

$$\mathscr{C} = [\mu' r^2 + \mu(ix + jy + kz) + Ix + Jy + Kz] \times [\mu' r^2 + \mu(ix + jy + kz) - (Ix + Jy + Kz)] = 0$$

and

$$\mathscr{C}' = [\mu' r^2 + \mu(ix + jy + kz)]^2 - [Ix + Jy + Kz]^2 + (\mu'^2 - \mu^2) r^2 (x^2 + y^2 + z^2 - r^2).$$

From the form of $\mathscr{C}$ we obtain:

THEOREM XI. *When the stop is at the point of contact of a plane $\mathscr{A}$ with the object space aplanatic surface, two planes $\mathscr{P}_1$, $\mathscr{P}_2$ will have a conicoid as their image provided*

the three planes $\mathscr{A}$, $\mathscr{P}_1$, $\mathscr{P}_2$ have a common axis of intersection, and any straight line through C, the centre of the refracting sphere, is divided harmonically at C and at its intersections with the planes $\mathscr{P}_1$, $\mathscr{A}$, $\mathscr{P}_2$.

When the surfaces are referred to axes parallel to their principal axes the equation for $\mathscr{C}$ becomes

$$\mathscr{C} = \mathscr{P}_1\mathscr{P}_2 = \left(\mu ix + \mu\epsilon jy + \frac{\mu' r^2\epsilon}{1+\epsilon}\right)\left(\mu ix - \mu\epsilon jy - \frac{\mu' r^2\epsilon}{1-\epsilon}\right) = 0,$$

the line of intersection of $\mathscr{A}$, $\mathscr{P}_1$, $\mathscr{P}_2$ being

$$\mu ix = -\mu\epsilon^2 jy = \frac{\mu' r^2\epsilon^2}{1-\epsilon^2}.$$

The equation of the image surface is

$$A(Bx + \mu\mu' i)^2 + B(Ay + \mu\mu' j)^2 + (\mu'^2 - \mu^2)\,ABz^2 = \mu^4(\mu'^2 - \mu^2)\,D^2\epsilon^2 r^2,$$

where

$$A = \mu'^2 - \mu^2 D\epsilon^2, \quad B = \mu'^2 - \mu^2 D, \quad D = (i^2 + j^2\epsilon^2)/r^2\epsilon^2.$$

Corresponding to any point on the line of intersection of $\mathscr{P}_1$ and $\mathscr{P}_2$ there are two image points, the magnifications m_1 and m_2 being connected by the relation

$$\frac{1}{m_1} + \frac{1}{m_2} = 2\frac{\mu'^2}{\mu^2}.$$

This result, however, is only of mathematical interest, since the circumstances they imply either do not arise or are subject to very severe limitations. If $\mu' > \mu$ the incident ray does not meet the refracting surface at all, and if $\mu' < \mu$ the conditions assumed only hold for object points within the refracting sphere.

Other stop positions

If now we turn to consider other stop positions a complication arises through the image of the stop itself being subject to aberration. The effect of this aberration is often small, but it prevents the adoption of an elementary treatment, and so is not considered here. The cases that we have discussed enable us, however, to trace the variations in the image of a surface, for example a plane, when the aperture is large by taking a series of elementary apertures distributed over the lens surface. The image for every elementary aperture is a portion of a conicoid, and good definition will only be attained when the corresponding parts of these conicoids are coincident or nearly coincident. By treating the imagery due to the surface in this way some indication will at least be gained of the extent to which reliance can be placed upon the usual elementary theory in experimental work.

Lenses

In practice we are more interested in the properties of lenses than in those of single surfaces. It is thus of interest to consider whether corresponding results can be established for lenses. From the elementary cases we have studied we can only draw conclusions for systems which have a suitable position for a stop common to both surfaces. Apart from some other systems of interest we can clearly include the whole class of thin lenses. It is at once evident that with these lenses groups of

conicoids can be found which have conicoids as their images. This will follow for example if we take as an object, supposed to lie in the medium of which the lens is made, any plane surface. The image of any plane in either refracting surface is a conicoid, and these are therefore conjugate to one another. A direct investigation can readily be made. It will suffice to consider a thin lens with the stop at the common vertex of the two surfaces. The ordinary secondary equation is equivalent to

$$\frac{1}{q}-\frac{1}{p}=\frac{\mu'\cos\phi'-\mu\cos\phi}{(\mu'-\mu)f},$$

where f is the focal length for the normally incident ray, μ' is the refractive index for the material of the lens, and μ that of the surrounding medium in which f is measured. It follows immediately that the image of

$$\mathscr{C}=[\alpha x+\beta y+\gamma z+(\mu'-\mu)f]^2-\delta^2[(\mu'^2-\mu^2)(x^2+y^2+z^2)+\mu^2x^2]=0$$

is

$$\mathscr{C}'=[(\alpha+\mu)x'+\beta y'+\gamma z'+(\mu'-\mu)f]^2 - (\delta+1)^2[(\mu'^2-\mu^2)(x'^2+y'^2+z'^2)+\mu^2x'^2]=0,$$

where the axis of the lens coincides with the axes of x and x'. In discussing these equations we will begin with the simpler cases. Putting first $\alpha-\delta\mu=\beta=\gamma=0$ we obtain:

THEOREM XII. *When the stop is situated at the vertex of a thin lens there is a system of object spheres which have spherical images. The centres of all these spheres lie on the axis of the lens, and the magnification corresponding to any sphere is constant* $\left(m=\frac{\delta}{\delta+1}\right)$. *Special examples of importance are the two principal focal spheres* $\mathscr{F}$ *and* $\mathscr{F}'$, *which are conjugate to the spheres at infinity* ($\mathscr{F}$ *belongs to the object space, and* $\mathscr{F}'$ *to the image space*). *The radii of* $\mathscr{F}$ *and* $\mathscr{F}'$ *are each equal to* $\frac{\mu' f}{\mu+\mu'}$, *their centres are distant* $\frac{\mu f}{\mu+\mu'}$ *from the vertex of the lens, and they meet the plane of the stop in a circle of radius* $f\left(\frac{\mu'-\mu}{\mu'+\mu}\right)^{\frac{1}{2}}$.

Changing the sign of α we have:

THEOREM XIII. *There exists a second series of object spheres (the reflection images of the previous series in a plane mirror normal to the axis through the lens vertex) which have non-spherical conicoids of revolution for their images. The images of corresponding spheres of both series meet the plane of the stop in the same circle.*

Putting $\beta=\gamma=\delta=0$:

THEOREM XIV. *When the stop is situated at the vertex of a thin lens the image of any plane normal to the axis of the lens is a conicoid of revolution with the axis of revolution along the lens axis. The semi-latus-rectum of the generating conic is equal to* $\frac{\mu' f}{\mu+\mu'}$, *so that all these image surfaces are of equal curvature at their vertices. If the object plane*

does not meet the focal sphere $\mathscr{F}$ the image is an ellipsoid; if it touches it the image is a paraboloid, and if it cuts it an hyperboloid of two sheets, the asymptotic cone being parallel to the cone through the intersection of the plane with $\mathscr{F}$ which has its vertex at the centre of the stop.

When $\gamma = \delta = 0$ we obtain:

THEOREM XV. *When the stop is situated at the vertex of a thin lens the image of any plane not normal to the axis of the lens is a conicoid with the plane ($z = 0$) through the lens axis and the normal to the object plane from the lens vertex as one of its principal sections. The polar plane with respect to the image conicoid of the point in which this principal section is met by the line of intersection of the object plane and the normal stop plane ($x = 0$) is the plane ($y = 0$) through the lens axis normal to the principal section. The radius of curvature of the conicoid in this polar plane where it is met by the lens axis is $\frac{\mu' f}{\mu + \mu'}$. In the principal section through the lens axis the radius of curvature at these points is $\frac{\mu' f}{\mu + \mu'} \sec^3 \theta$, where θ is the angle between the normal to the surface and the lens axis. The polar plane of the lens vertex, the normal plane through the stop, and the object plane have a common line of intersection. The three axes of the conicoid are unequal and no axis is normal to the object plane.*

The general conclusions when $\beta = \gamma = 0$ are stated in the next theorem.

THEOREM XVI. *When the stop is situated at the vertex of a thin lens, a conicoid of revolution with its major axis coincident with the axis of the lens will have as its image another conicoid of revolution also with its major axis coincident with the lens axis, provided the transverse semi-axis b and the distances x_1 and x_2 of the vertices of the object conicoid are connected by the relation*

$$(\mu'^2 - \mu^2) b^2 + \mu'^2 x_1 x_2 = 0.$$

The corresponding relation for the image conicoid is then satisfied, and the difference in curvature of the object and image surfaces on the axis is $(\mu' + \mu)/\mu' f$.

There is in no case any loss of generality involved in putting $\gamma = 0$. From this simplified equation the following results are derived:

THEOREM XVII. *When the stop is situated at the vertex of a thin lens of refractive index μ' immersed in a medium of index μ the conditions which must be satisfied for a conicoid $\mathscr{C}$ to have another conicoid $\mathscr{C}'$ for its image are:*

(i) *the section of $\mathscr{C}$ by the plane of the stop ($x = 0$) is a conic of which the lens vertex is the focus: the directrix of this section will be denoted by $\mathscr{D}$;*

(ii) *the plane ($z = 0$) through the lens axis and perpendicular to $\mathscr{D}$ is a principal section of the conicoid;*

(iii) *the point of intersection of this principal section with $\mathscr{D}$ is the pole of the plane ($y = 0$) through the lens axis normal to the principal section;*

(iv) *the transverse semi-diameter c of the section of $\mathscr{C}$ by this plane ($y = 0$) is*

connected with the distances x_1, x_2 of the lens vertex from the points of intersection of the lens axis with $\mathscr{C}$ by the relation

$$\frac{c^2}{x_1 x_2} = \frac{\mu'^2}{\mu'^2 - \mu^2}.$$

(*An alternative way of stating the same condition is that the radius of curvature ρ of this section at the points on the lens axis is related to z_0, the distance from the lens vertex of the intersection of $\mathscr{C}$ with the stop plane measured in this section, by the condition*

$$\frac{\rho^2}{z_0^2} = \frac{\mu'^2}{\mu'^2 - \mu^2}.)$$

The conjugate conicoid $\mathscr{C}'$ is specified by corresponding conditions, the directrix $\mathscr{D}$ being common to $\mathscr{C}$ and $\mathscr{C}'$, and by the further conditions that

(i) *the curvatures on their major axes of the conic sections made by the intersection of the stop plane ($x = 0$) with $\mathscr{C}$ and $\mathscr{C}'$ differ by* $\frac{1}{f}\left(\frac{\mu' + \mu}{\mu' - \mu}\right)^{\frac{1}{2}}$;

(ii) *the curvatures of $\mathscr{C}$ and $\mathscr{C}'$ in the axial plane parallel to $\mathscr{D}$ differ by* $\frac{\mu + \mu'}{\mu' f}$.

In considering the relation between the curvatures in the plane $x = 0$ it must be remembered that under the conditions assumed this plane is not self-conjugate.

General curvature relations

In conclusion we may note two general results it is easy to establish connecting the curvatures of axial sections of any conjugate surfaces, whether primary or secondary, formed by light refracted at the common vertex of a series of thin lenses in contact. They depend upon the general theorem that if r, ϕ are the polar coordinates of any plane curve

$$\frac{1}{r} + \frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) = S \operatorname{cosec}^3\theta,$$

where S is the curvature at (r, ϕ) and θ is the angle the curve makes with the radius vector. This angle is given by the equation

$$\frac{1}{r}\cot\theta + \frac{d}{d\phi}\left(\frac{1}{r}\right) = 0.$$

The secondary equations for conjugate points

$$\frac{1}{q} - \frac{1}{p} = \Sigma a\,(\mu' \cos\phi' - \mu\cos\phi), \quad \mu' \sin\phi' = \mu \sin\phi$$

are clearly suitable for the application of these equations. We have, on denoting $\mu' \cos\phi' - \mu\cos\phi$ for brevity by u,

$$\frac{du}{d\phi} = u\tan\phi',$$

$$\frac{d^2u}{d\phi^2} = u\tan^2\phi' + u\sec^2\phi' \cdot \frac{\mu\cos\phi}{\mu'\cos\phi'},$$

so that

$$u + \frac{d^2u}{d\phi^2} = u\frac{\sec^3\phi'}{\mu'}\,(\mu'\cos\phi' + \mu\cos\phi)$$

$$= \frac{\mu'^2 - \mu^2}{\mu'}\sec^3\phi'.$$

We therefore have the conclusion:

THEOREM XVIII. *In plane sections of conjugate secondary surfaces made by a plane through the axis of a number of thin lenses in contact, if the inclinations of these curves with the ray through the vertex of the surfaces (where a small stop is situated) making an angle ϕ with the axis are θ, θ', their curvatures S, S' are connected by the relation*

$$S' \operatorname{cosec}^3\theta' - S \operatorname{cosec}^3\theta = \Sigma\kappa \frac{\mu' + \mu}{\mu\mu'} \sec^3\phi',$$

where κ is the paraxial power of a typical component lens of refractive index μ', μ is the refractive index of the medium in which the lenses are immersed, and ϕ' is the angle made with the axis by the light in traversing that lens.

If we suppose $\mu = 1$, and write ω for the reciprocal of μ', the sum on the right of this curvature equation is

$$\Sigma\kappa (1 + \omega) (1 - \omega^2 \sin^2\phi)^{-\frac{3}{2}}.$$

This shows that all sums of the form

$$\Sigma\kappa\omega^n$$

or better

$$\Sigma R (1 - \omega^{2n-1})$$

for positive integral values of n are important quantities in expressions for the properties, including aberrations, of such lenses. The power of the combination is the particular case $n = 0$, and the Airy-Petzval sum is obtained when $n = 1$.

Turning now to the primary equation

$$\frac{1}{q} - \frac{1}{p} = \sec^2\phi \Sigma\, a (\mu' \cos\phi' - \mu \cos\phi),$$

put

$$v = \sec^2\phi (\mu' \cos\phi' - \mu \cos\phi).$$

Then

$$\frac{dv}{d\phi} = v (\tan\phi' + 2 \tan\phi),$$

$$\frac{d^2v}{d\phi^2} = v \left[(\tan\phi' + 2 \tan\phi)^2 + \sec^2\phi' . \frac{\mu \cos\phi}{\mu' \cos\phi'} + 2 \sec^2\phi \right],$$

so that

$$v + \frac{d^2v}{d\phi^2} = \frac{v}{\mu'} \sec\phi' (\sec^2\phi' + 4 \tan^2\phi) (\mu' \cos\phi' + \mu \cos\phi) + 2v \sec^2\phi.$$

This leads to

THEOREM XIX. *In plane sections of conjugate primary surfaces made by a plane through the axis of a number of thin lenses in contact, if the inclinations of these curves with the ray through the vertex of the surfaces (where a small stop is situated) making an angle ϕ with the axis are θ, θ', their curvatures P, P' are connected by the relation*

$$P' \operatorname{cosec}^3\theta' - P \operatorname{cosec}^3\theta = \sec^2\phi \Sigma\kappa \frac{\mu' + \mu}{\mu\mu'} \sec\phi' (\sec^2\phi' + 4 \tan^2\phi) + 2 \sec^4\phi \Sigma \frac{K}{\mu},$$

where κ is the paraxial power of a typical component lens of refractive index μ', μ is the refractive index of the medium in which the lenses are immersed, ϕ' is the angle made with the axis by the light in traversing the lens and K is the power of a typical component for light travelling in this direction.

Since, when $\mu = 1$, $\omega\mu' = 1$,

$$\sec^2\phi' + 4\tan^2\phi = \sec^2\phi\sec^2\phi'\,(1 + 3\sin^2\phi - 4\omega^2\sin^4\phi)$$

and

$$\frac{K}{\kappa} = \frac{(1-\omega^2\sin^2\phi)^{\frac{1}{2}} - \omega\,(1-\sin^2\phi)}{1-\omega},$$

the primary sum may be expressed in terms of the same sums as occur in the expansion of the secondary sum. This is otherwise evident, since both relations have been derived from ΣK by differentiation with respect to ϕ.

The results which have been given are by no means exhaustive of the conclusions which can be reached with conjugate surfaces of the second order, but they are probably those of most general interest for comparatively elementary work. They suffice to show that the consideration of images of surfaces of the second order is important and interesting. At the lowest estimate the inclusion of a few examples of such imagery in the course will enable the pupil to realise that the usual theory is of extremely limited value, and that the results to which it leads are propositions about the tangent planes to the image surfaces on the lens axis, and do not relate to the surfaces themselves. It may serve the further useful end of stimulating interest in the study of second order surfaces among pupils who have not already realised their great practical importance.

DISCUSSION

Capt. T. Y. Baker: I quite agree with Mr Smith that in teaching Geometrical Optics the student should not be led into wrong beliefs by inexact statements as to the nature of optical imagery, but the fact remains that in practically all optical instruments Gaussian image formation is the final result that the designer attempts to achieve and it is therefore natural for the teacher to lay stress upon the approximations that lead to the equations of Gaussian imagery. The student's fundamental conception of the formation of an optical image is, nine times out of ten, derived from his acquaintance with a photographic camera. He knows that for distant objects the image is formed on a plane in a certain position. As the object is brought nearer, the "image plane" is further from the lens but it is still a plane in accordance with his experience and I do not think that any serious harm is done to his acquisition of the laws of image formation by the limitation to small fields and small apertures that the teacher has to impose in order to develop the Gaussian laws.

It would seem from a perusal of Mr Smith's paper that he would prefer that the student should be taught that the image of a plane is not a plane but is some form of curved surface—or rather surfaces, the primary and the secondary—and he even seems to think it almost immoral to suggest that a workable approximation is to take the tangent plane at the vertex of the surface as representing the surface itself. He would prefer to deal with the various conicoids, etc. that he derives from the algebraic investigation of the formula given on p. 131. But is he not himself making quite as big a misstatement in so doing? Take any narrow pencil originating

from a point in his object plane and consider what happens to it when it is refracted into another medium. That pencil does not form a point image at all. The spherical wave-front is refracted into a non-spherical one and the best that can be said of the image formation is that the rays all pass through the two focal lines, *on the assumption that the pencil is narrow*. Given that assumption, the locus of the position of the secondary focus for movement of the object point over a plane will be one of the surfaces investigated in the paper, but the student is no nearer to forming any accurate conception than he was before as to what has happened to the light. He has been taught that as long as there is the restriction to small fields and small apertures the position and size of the image follow Gaussian laws. Mr Smith seems to want to replace that teaching by something like this: "As long as you keep the apertures small, but not the fields, then the secondary image of a plane will not be a plane, but, in certain circumstances, a conicoid." Is this really any great advance towards a realisation of what happens in a practical instrument where neither the field nor the aperture is small? Personally I do not think it is and I feel sure that Mr Smith, if he were to try to teach optics in this manner to a class of students, would soon find himself in difficulties. May I put myself in the position of the "over-alert or unusually obstinate pupil" and ask him how he is going to deal with spherical aberration when he comes to it? He will have to come to it some time and, when he does, he will be compelled to abandon his conception of an optical system in which the aperture is small. Without that restriction the present paper would be meaningless. Without that restriction the image of a point on the axis is not a point at all. The bundle of convergent light narrows down to a waist, and any position near that waist is the best that can be done in the way of focussing. Without that restriction Mr Smith will have to admit that there is, in practice, no such thing as a secondary conjugate *surface*; that the image of a plane is not a surface at all but rather a *space* within which the bundles of light are more nearly in focus than anywhere else.

Let us put the matter in another way. Suppose we obtain formulae for the magnitudes of the first order aberrations, say for a thin lens, with a stop of finite aperture, either from ray equations or from the characteristic function. From these formulae can be deduced the necessary conditions for elimination of first order aberration. With a certain stop position and a certain lens shape there will be no spherical aberration; with other values there will be no coma; with a third set there will be no astigmatism and the field curvature will be reduced to that of the Petzval curvature. Vary the shape of the lens, vary the position of the stop, and the spherical aberration will change, coma will change, curvatures of sagittal and tangential fields will change. The only thing that remains constant among the five von Seidel defects is the Petzval curvature of field. I put it in this way because I want to make it clear that curvature of the secondary image, which is mainly what Mr Smith's paper is about, is, quite logically, a first order aberration and should be treated as such. Mr Smith, as a mathematician, has no right to say that Gaussian image formation, which is based on the second degree terms of the characteristic function, is untrue because it leads to the conclusion that the image

of a plane is another plane. He would be quite justified in saying that, if the next order terms of the characteristic function are included, then the image of a plane is no longer a plane. But he has not said that in his paper, or even implied it. He has dealt with the shape of the secondary surface, i.e. he has really dealt with the coefficient of astigmatism, certainly in the absence of spherical aberration, and I think also in the absence of coma. The plane fields of Gaussian imagery are at least logical representations of what happens in the simplest approximation to the characteristic function. Mr Smith goes to the terms of the next order, suppresses some of them, and serves the remainder up as a set of conicoids of revolution although, in reality, he is not entitled, in an axial system, to say anything more about the surfaces than relates to their curvatures at their vertices. If he wants to do more he must expand the characteristic function further. I sympathise with him in his desire to explain optical imagery to students in a manner more in accordance with reality than the Gaussian equations involve, but surely he is not entitled to derive image forms from a part only of the next order terms and then say "This is truth; the Gaussian image is a lie."

I strongly suspect that he has taken a simple formula for thin pencils and, forgetting the limitation imposed on that formula, has used it as a peg upon which to hang a lot of nicely reasoned analysis of conicoids and has pushed that analysis much further than the formula warrants. As a piece of three-dimensional analysis it is interesting and elegant, but it is not optics and it is not the way to teach optics.

Mr Smith: Capt. Baker's remarks are vigorous and entertaining. I cannot hope in a brief reply to deal with them in detail: if I attempted it my difficulty would be increased by the rapid rate of change the meanings of some of the more important words appear to suffer in the course of his comments. I labour under the further handicap of not knowing precisely what Capt. Baker means by Gaussian image formation, Gaussian laws, etc. I infer that he applies the label to a rather different group of concepts from those to which I should apply it. The two groups undoubtedly overlap, but as I do not know the extent of the overlap I will endeavour to avoid misunderstanding by omitting Gauss's name.

Much of what Capt. Baker has to say rests on a complete misapprehension of the underlying theory. I can only regret that I have failed to make the basis quite clear. In effect it is said that the work treats a minor part of an approximation as though it were the whole and were free from error. Actually the results given are exact over the whole field of view; no approximations are introduced at any stage. The only relations used are the exact law of refraction, and the exact collinearity of any pair of conjugate secondary points and the centre of curvature of the refracting surface.

One aspect of this criticism is the suggestion that the method is of no practical value because it will have to be abandoned when spherical aberration, coma, and so on come to be considered. It is hardly possible to treat very seriously an objection which rests on the ground that the whole subject of geometrical optics is not exhausted in a single paper. No attempt has been made to deal specifically with

these aberrations. The implication in the criticism is however false. Analysis into distinct aberrations is quite arbitrary, and all the traditional aberrations can be derived from the changes in the secondary surface as the effective stop moves over the whole aperture as readily as by the alternative ways described in some textbooks. In this respect the method compares favourably with the treatment Capt. Baker prefers, which leads only to a dead end, so that a fresh start has to be made before any further progress can be made.

I hope it is clear from the paper that I am not objecting to the early instruction pupils now receive: my objection is to presenting the collinear theory without any subsequent corrective indicating the importance of the unavoidable departures from collinear correspondence.

As I know that Capt. Baker is not alone in holding the views stated in his first paragraph I should like to say distinctly that they are erroneous. The fact that a plane object has a plane image does not mean that the imagery is of the type Capt. Baker implies. This kind of imagery is not in fact attainable with a photographic lens. It would require the law of refraction to be

$$\mu' \tan \phi' = \mu \tan \phi.$$

Apparently Capt. Baker is satisfied with this law. On the other hand I have attempted to build on the law

$$\mu' \sin \phi' = \mu \sin \phi.$$

On the consequences of this preference Capt. Baker comments "it is not optics and it is not the way to teach optics." I am interested in the opinion, but I do not share it.

GRAPHICAL CONSTRUCTIONS FOR A REFRACTED RAY

By T. SMITH, M.A., F.Inst.P.

(Optics Department, The National Physical Laboratory.)

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ABSTRACT. Two groups of spherical or plane conjugate surfaces can be found for refraction at any point of a sphere, and they give rise to distinct methods of constructing a refracted ray. Snell's construction is a special case of one group, and Young's a special case of the other group. Other cases of interest are examined. By inverting these two groups other constructions are obtained.

Two or three methods of constructing the refracted ray corresponding to a given incident ray, involving the use only of ruler and compasses, are described in text-books on optics. They accomplish all that is necessary. One of them has recently been admirably adapted to the requirements of draughtsmen by J. H. Dowell*. The new methods to be described achieve nothing which is not equally achieved by the old ones. The justification for their presentation is the belief that, as among philosophers of a bygone age, so also among us curiosity is no less a commendation than utility.

So far as I am aware almost all methods depend upon the determination of conjugate secondary points for an artificial system which touches the real system at the point of refraction. In another paper† I have expressed the general equations of conjugate secondary surfaces for refraction at a sphere in the form

$$p = \mu a w, \quad q = \mu' a w',$$

where p and q are the distances of the conjugate points from the point of refraction, μ and μ' are the two refractive indices, w and w' are the distances of the conjugate points from the centre C of the refracting sphere, and a is any number, in general depending on the direction of the line through the centre C and the two conjugate points. It is convenient to assume throughout that $\mu' > \mu$.

If we suppose that a is a constant, the two surfaces are spheres. It is important to realise that though one sphere is the image of the other the magnification varies from point to point of the surface. The corresponding plane construction is shown in Fig. 1. R is the point of refraction, and P and Q two conjugate points, so that P, Q, C are collinear, $RP = p$, $RQ = q$, $PC = w$, $QC = w'$. The equations then mean that P lies on the circle on AA' as diameter, where A and A' are the points in

* "Graphical Methods Applied to the Design of Optical Systems," *Proc. Opt. Conv.* 1926 Part II (965).

† "Secondary Conjugate Surfaces," *Trans. Opt. Soc.* **32** 1930–31 (129).

which RC is divided internally and externally in the ratio $\mu a : 1$. Similarly Q lies on the circle on BB' as diameter, where RC is divided internally and externally in the ratio $\mu' a : 1$ at B and B'. When R, C and the two circles are drawn the refracted ray RQ corresponding to the incident ray RP is found by joining P, a point of intersection of the incident ray with the corresponding circle APA', to C, the join meeting the second circle in Q. It should be observed that either point of intersection of the ray with the circle may be taken. A feature of interest in this construction is that if we replace a by $1/a\mu\mu'$ the two circles are interchanged and

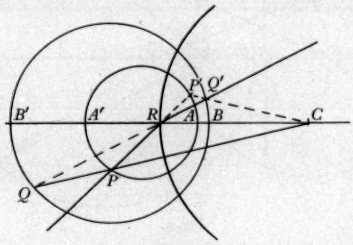

Fig. 1.

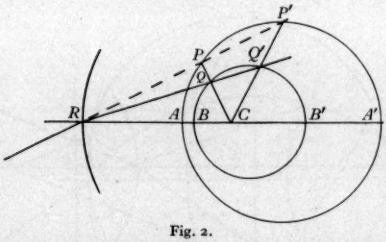

Fig. 2.

arranged about C in a manner corresponding to their previous disposition about R. Fig. 2 shows the construction complementary to Fig. 1.

Snell's is a particular case of this construction, viz. the case when C is infinitely distant. The two circles then have R as a common centre, and PQ is parallel to the normal RC.

By selecting different values of a we change the relative sizes of the two circles. Special cases of interest are

(i) $\mu a = 1$; the object sphere degenerates into the plane bisecting RC at right angles. The corresponding construction is shown in Fig. 3. The relative dimensions are given by

$$\frac{2.RA}{\mu'^2 - \mu^2} = \frac{2.AC}{\mu'^2 - \mu^2} = \frac{BC}{\mu(\mu' - \mu)} = \frac{CB'}{\mu(\mu' + \mu)} = \frac{BF}{\mu\mu'} = \frac{CF}{\mu^2}.$$

The plane shown in the diagram is only a part of an infinite sphere. The rest is at infinity. We thus obtain an alternative construction by drawing a parallel through C to the incident ray, meeting the circle in Q'. RQ' is then the refracted ray.

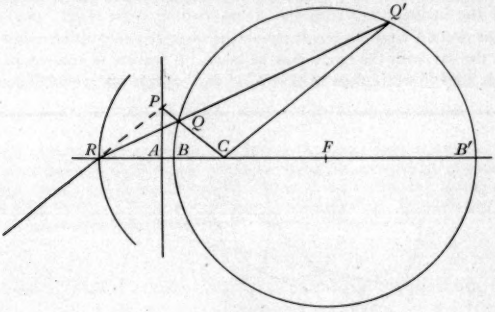

Fig. 3.

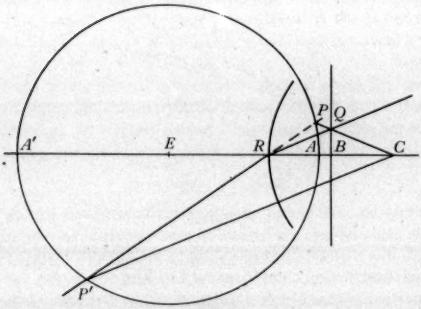

Fig. 4.

(ii) $\mu' a = 1$; this is the complement to the previous example: the object surface is spherical and its image is plane. The relative dimensions for the construction shown in Fig. 4 are

$$\frac{2.RB}{\mu'^2 - \mu^2} = \frac{2.BC}{\mu'^2 - \mu^2} = \frac{A'C}{\mu'(\mu' + \mu)} = \frac{AC}{\mu'(\mu' - \mu)} = \frac{EA}{\mu\mu'} = \frac{EC}{\mu'^2}.$$

(iii) $(\mu\mu')^{\frac{1}{2}} a = 1$. The object and image spheres are equal, and are similarly

situated with respect to R and C. This is thus the self-complementary case. The dimensions of this case, illustrated in Fig. 5, are given by

$$\frac{ER}{\mu} = \frac{CF}{\mu} = \frac{RC}{\mu' - \mu} = \frac{EA}{(\mu\mu')^{\frac{1}{2}}} = \frac{BF}{(\mu\mu')^{\frac{1}{2}}} = \frac{AB}{(\mu'^{\frac{1}{2}} - \mu^{\frac{1}{2}})^2}.$$

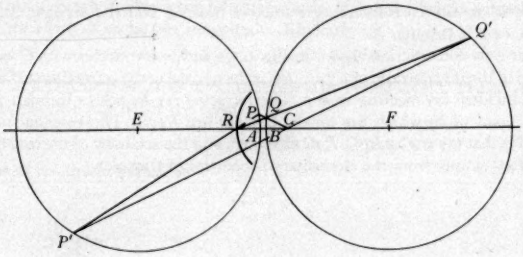

Fig. 5.

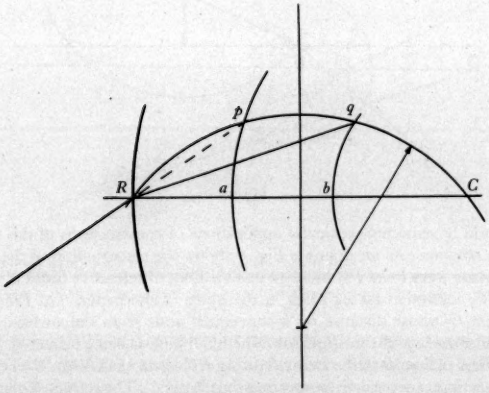

Fig. 6.

The form of the ratio involving AB illustrates the general rule that these object and image surfaces do not intersect one another.

We may observe that when, as in the last example, the two circles are described in opposite directions as the incident ray changes, their radii may bear any arbi-

trarily assigned ratio to one another. When they are described in the same direction the ratio of the radii may not lie between the values μ/μ' and μ'/μ.

We obtain a different type of construction by inverting the general proposition from which we set out with respect to a circle centered at R. The circles APA', BQB' (Fig. 1) invert into concentric circles with C as centre and their radii inversely proportional to the corresponding refractive indices, and the straight line PQC becomes a circle through R.

In the new construction shown in Fig. 6, ap and bq are circles with C as centre and radii in the inverse ratio of $\mu : \mu'$. RC is the normal to the refracting surface at R, Rp is an incident ray meeting ap at p. The refracted ray Rq passes through q where q is the point of bq which lies on the circular arc $RpqC$. The essential property involved is that the angles RpC, RqC are equal, and the accuracy of the construction is seen very simply from the elementary properties of triangles.

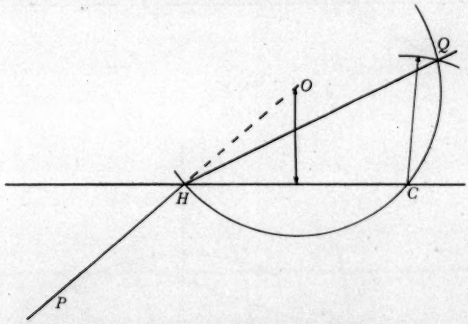

Fig. 7.

As would be expected particular applications of constructions of this group are somewhat obvious. As an example Fig. 7 shows the construction of the refracted part of a plane wave front PH incident in a medium of refractive index μ on a plane surface HC, refraction taking place in the plane of the paper. On PH produced take a point O whose distance on a convenient scale from the surface HC is μ', the refractive index of the medium into which the wave is being refracted. With O as centre, OH as radius, describe a circle cutting HC again in C. With C as centre and 2μ as radius draw a second circle meeting the first in Q. The refracted plane wave is represented by HQ.

We now consider constructions of the group to which Young's method belongs. The spherical conjugate surfaces to be considered in this group are surfaces of constant magnification m. The condition to be satisfied is $w'/w = m = \text{constant}$. It is apparent from this condition that a is no longer a constant quantity.

The equations we require are obtained by eliminating a when the characteristic

condition is satisfied. For any given direction, and therefore for a definite value of a, the equation

$$q = a\mu' w' = a\mu' mw$$

or

$$q' = \frac{q}{m} = a\mu' w$$

means that q' is what q becomes when the scale of length shrinks symmetrically about C to $1/m$th of its former magnitude. Similarly

$$p' = mp = a\mu mw = a\mu w'$$

means that p' is what p becomes when the scale is increased m fold. The equation for the object surface now takes the form

$$p = \frac{\mu}{\mu'} q',$$

and that of the image surface is

$$q = \frac{\mu'}{\mu} p'.$$

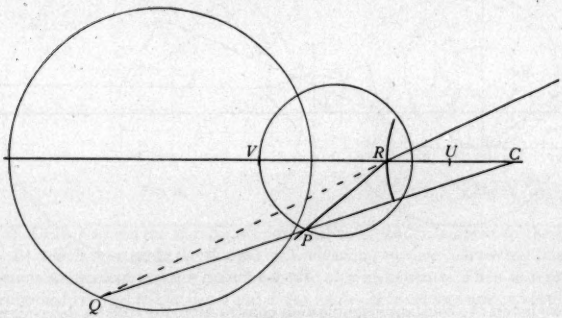

Fig. 8.

In other words, if U and V are points on RC such that

$$\frac{RC}{UC} = m = \frac{VC}{RC},$$

the object surface is the sphere which divides RU internally and externally in the ratio $\mu : \mu'$, and the image surface is the sphere which divides RV internally and externally in the ratio $\mu' : \mu$. Fig. 8 shows the result for $m = 2$, $\mu'/\mu = 1{\cdot}5$.

From this description it is clear that the constant magnification spheres form a distinct group of conjugate surfaces from those already described. C is clearly a centre of similitude for any pair of the new surfaces, and a pair can only be concentric when the common centre is at C. It is easy to show from the relations given that this only occurs when $m = \mu'^2/\mu^2$; the resulting surfaces are Young's aplanatic spheres, and the corresponding construction is well known.

The constructions of this group are less symmetrical than those considered previously. One manifestation of the lower symmetry appears in the need for selecting the proper points of intersection of the ray and sphere instead of being at liberty to take either.

A convenient special case for graphical work arises when $m = 0$. The lengths RF, BF, CF are set off along the normal in the ratio $\mu'^2 : \mu'\mu : \mu^2$. With F as centre a circle is struck with FB as radius, meeting the parallel to the incident ray through C in Q'. RQ' is the refracted ray. It will be observed that these three lengths are proportional to the three radii involved in Young's construction, but the new diagram is more compact because the three lengths are all measured in the same direction from F instead of the mean being measured in the direction opposite to the extreme lengths as in Young's method. This is the same construction as the second method illustrated in Fig. 3.

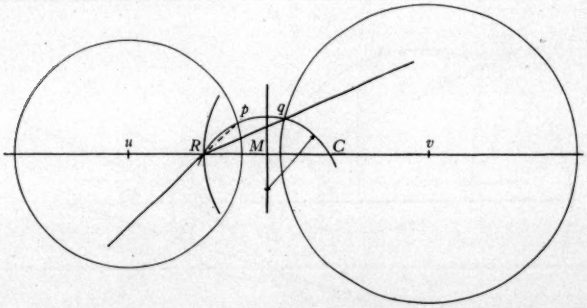

Fig. 9.

If we invert the constant magnification construction, the spheres determined by RU and RV respectively become spheres having centres at u and v where M, the mid-point of RC, is also the mid-point of uv. The radii of these spheres are respectively $\frac{\mu'}{\mu}.Ru$ and $\frac{\mu}{\mu'}.Rv$. In Fig. 9 these are represented by circles. If an incident ray Rp meets the former in p the emergent ray is Rq, where q is the point of intersection of the circle with v as centre and the circular arc RpC.

DISCUSSION

Mr A. Whitwell (communicated): I have read Mr Smith's paper with very great pleasure. In it he shows the relation between several isolated constructions familiar to many of us; it reminds me of two short articles I wrote to the *Optician* (2nd Feb. and 1st June, 1923) in which I described three constructions devised by myself. One of these is included in Fig. 3 of Mr Smith's paper. The other two, described

below, may be of interest to some readers who may not have seen them and I have no doubt Mr Smith will correlate them and put them in one or other of his groups.

(1) Draw two circles of radii μ'/r and μ/r tangent at the point x (Fig. A). On the line through the centres mark a point o at a distance μ'/r from the centre of the smaller circle. From the point o draw any line oa to represent an incident ray making an angle of incidence $xoa = \theta$ with the line ox. With centre o and radius oa describe the arc ab. The line ob will represent the refracted ray and the angle $xob = \theta'$ the angle of refraction. Incidentally, the length

$$oa = ob = \frac{\mu' \cos\theta' - \mu \cos\theta}{r}$$

will represent the secondary power in dioptres if r be measured in metres.

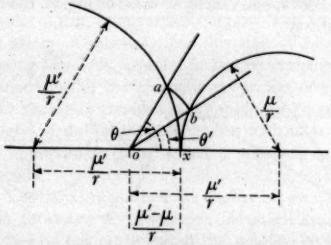

Fig. A.

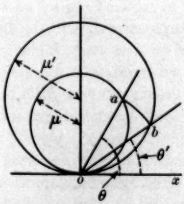

Fig. B.

(2) Draw two circles of radii proportional to μ' and μ tangent to the line ox (Fig. B) which represents the normal to a refracting surface. Draw any line oa to represent an incident ray and describe an arc ab with centre o. Then ob represents the refracted ray and the angles θ and θ' the angles of incidence and refraction.

Mr Smith: I am much indebted to Mr Whitwell for drawing my attention to the two articles he has mentioned. I had not seen them before; they are full of interest and should be consulted. The construction shown in Fig. 3 to which Mr Whitwell refers is that obtained by drawing CQ' parallel to the incident ray. In one of his papers Mr Whitwell uses this very effectively to construct the caustic for a parallel beam derived from an axial point, i.e. to determine completely the spherical aberration. This furnishes an excellent illustration of the way in which the secondary image surfaces may be used to give all other aberrations. Incidentally this is probably one of the simplest ways of plotting the refracted beam.

The two constructions Mr Whitwell illustrates are not included in either of the two groups I have described. This is evident if we note that all the constructions included in my paper have the properties that the centres of both the circles lie on the normal and that the circles do not meet or touch. Fig. A is obtained by constructing two triangles each having a side representing a given multiple of

$\mu' \cos \phi' - \mu \cos \phi$, and Fig. B by constructing two sides representing $\mu \sin \phi$ or its equivalent $\mu' \sin \phi'$. It is of interest to consider the relations of these two constructions to the older constructions.

If we invert the ordinary form of Snell's construction we obtain the refracted ray by the construction shown in Fig. C. PQR is any circle touching the normal PN at the point of refraction P. If the incident ray PQ cuts the circle again in Q the refracted ray meets the circle again at R, where $\mu PQ = \mu' PR$. Fig. B is the inverse of the modified Snell's construction shown in Fig. D. AQ and BR are parallel to the normal PN and their distances from it are inversely proportional to the refractive indices μ, μ', so that if the incident ray meets AQ in Q the refracted ray will cut BR in R, where $PQ = PR$. If we draw AQ and BR on opposite sides of the normal (as is more convenient) we modify Fig. B by having the two circles touching externally. The circles then correspond exactly to those of Fig. A, though they are there used in a very different way.

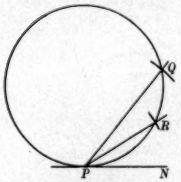

Fig. C.

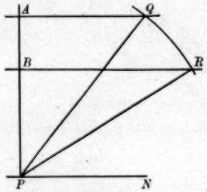

Fig. D.

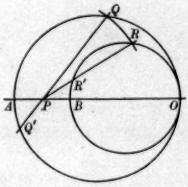

Fig. E.

We may note that if the incident and refracted rays in Fig. A meet the associated circles again in a' and b', $oa' = ob'$, and measure $\mu' \cos \phi' + \mu \cos \phi$ on a suitable scale. The two lengths $\mu' \cos \phi' \pm \mu \cos \phi$ are also given by intercepts in some other constructions, for example Young's.

The inverse of Fig. A gives an alternative construction in which the circles are arranged as in Fig. B. It is shown in Fig. E. The circles OQA, ORB touch internally at O and have their radii in the ratio $\mu' : \mu$. P is the mid-point of AB on the normal OBA. If the incident ray PQ cuts the former circle in Q the refracted ray PR meets the second circle in R, where $PQ = PR$; each of these lengths is proportional to $\mu' \cos \phi' + \mu \cos \phi$. We also have $PQ' = PR'$, where Q' and R' are the second points of intersection of the rays with their circles, and these are proportional to $\mu' \cos \phi' - \mu \cos \phi$. The variations are obvious if we consider that one figure is derived from another by rotating the plane of one circle through two right angles relatively to the other about axes through either O or P normal to the diameters OA, OB.

OPTICAL EXPERIMENTS WITH ELECTRONS

ABSTRACT OF A LECTURE BY PROF. G. P. THOMSON, F.R.S.

Delivered, 14th May, 1931.

It is now seven years since L. de Broglie brought forward the view of the duality between waves and particles which is now almost universally accepted under the name of wave mechanics, and which represents one of the greatest advances in physics of this, or any other, century. The original treatment was a development of the theory of relativity, but this side of the theory can no longer be kept in its original form. It appears, in fact, that the requirements of relativity are closely connected with the magnetic properties of the electron which gave rise on the older theory to the idea of a "spinning electron" and were not considered by de Broglie. I do not propose to deal with these to-day, and shall give the theory in the approximate form, which is sufficient to explain the experiments I have to describe.

The basis of the whole is a duality between waves and particles which is common both to matter and radiation. Maxwell made optics a branch of electricity; if de Broglie has not reversed the relation he has at least shown both as different cases of a common principle, which is more like old-fashioned optics than old-fashioned electricity. The duality takes this form; any observable event is representable as the arrival or departure of a particle at or from a small region of space, but the laws which govern these events involve a quantity which is best thought of as the amplitude of a wave (possibly in multidimensional space). In the case of light this quantity is indeed the electric or magnetic vector of the Maxwell wave (it is indifferent which).

In the case of electrons it is the more elusive ψ which obeys also an equation of the type known to mathematicians as a wave equation. In general ψ is complex, for the equation is complex, so no direct physical meaning can be assigned to it. Its modulus $|\psi|$ is real and de Broglie gives as his "principle of interference" the statement that the chance of the presence of an electron at a given place and time is proportional to $|\psi|^2$. The analogous statement in optics is that the chance of a quantum of light appearing at a given place and time is proportional to the square of the amplitude of the Maxwell wave.

According to de Broglie's theory the wave-length associated with a free electron is $\lambda = h/mv$, where h is Planck's constant and mv the momentum. He enunciates therefore a "law of spectral distribution" according to which the chance of the presence of an electron with a given momentum is proportional to the square of the modulus of the Fourier component of the wave whose λ corresponds to the

given momentum. In optics the chance of the appearance of a quantum of energy W is proportional to the square of the Fourier component of the Maxwell wave of wave-length hc/W.

This simple correspondence between electron and quantum—ψ-wave and Maxwell wave—in the case of the free electron, prepares us for a close experimental analogy. In fact, we can repeat many optical experiments with electrons and get strikingly similar results. The chief differences are due to the smaller wave-length, which is usually less than that of X-rays, and the much smaller penetrating power. Davisson and Germer made experiments with electrons which are the analogy to the diffraction of X-rays by a single crystal, as in the Bragg method.

Other experiments have reproduced with cathode rays the diffraction of X-rays by a crystalline powder, and have verified de Broglie's law of wave-length with considerable accuracy. Some recent work with cathode rays and single crystals of copper and silver provides the electron analogy to the optical experiment in which two transmission diffraction gratings are superposed with their rulings inclined to each other. The etched surface of the single crystal is apparently covered with a number of small lumps, probably the material left between etching pits. The cathode rays strike the crystal at a small glancing angle to the main surface, and pass through the lumps, being diffracted by the atoms in them. If the thickness of the lump, in the direction in which the rays traverse it, is less than a certain amount, which for the angles of diffraction and electronic wave-lengths used is of the order 10^{-6} cm., the thickness has no influence on the pattern. The diffracting system is then equivalent to an arrangement of atoms in the plane normal to the rays, and this two-dimensional array is mathematically equivalent to the crossed gratings of the optical experiment, giving rise to an array of spectra which is reciprocally connected with the atomic array producing it.

The spectra, when received on a fluorescent screen, are bright enough to be shown to a small audience.

DEMONSTRATION OF A NEW TYPE OF VISUAL SPECTROPHOTOMETER

By H. BUCKLEY, M.Sc., F.Inst.P., and F. J. C. BROOKES
(Photometry Division, The National Physical Laboratory.)

Demonstrated, 13th November, 1930.

The instrument demonstrated is a new type of visual spectrophotometer which has been developed at the National Physical Laboratory and described in a recent number of the *Journal of Scientific Instruments**. The novel feature of the instrument is that there are no moving optical media in the instrument.

Two light sources consisting of gas-filled lamps behind ground glass screens are used. The photometric scale of the instrument is in terms of the voltage-intensity relation of one of the lamps at a standard wave-length in conjunction with rotating sector discs of known transmission. The voltage-intensity relation for other wave-lengths is calculated from that at the standard wave-length.

* *Journ. Scient. Instr.* **7** 1930 (305).

REVIEWS

The Structure of Crystals (Second Edition). By RALPH W. G. WYCKOFF. Pp. 497; 274 Figs. (New York: The Chemical Catalog Co., Inc., 1931.) Price $7.50 net.

A revised edition of this monograph has long been expected. In 1924, when the first edition appeared, the science of crystal structure analysis by X-ray methods was, comparatively speaking, in its infancy. It was, however, a lusty babe, and in the past seven years it has grown with bewildering rapidity. Then, the crystals whose structure had been completely determined were few and simple, now they are many and complicated. As the writer himself points out, the advances of the past few years have necessitated the rewriting of nearly every chapter.

The form of the first edition has been retained and closely followed, though an introductory chapter has been added. According to the American Chemical Society, under whose auspices the monograph has been published, one of the main functions of the book is to present the knowledge available upon the chosen topic in a readable form, intelligible to those whose activities may be along a wholly different line. That being so, one can imagine the average chemist wishing that the introductory chapter had been considerably extended, for it is the most readable part of the whole book, and it is the only part of the book in which the direct bearing of crystal structure analysis on chemical problems has been adequately pointed out. In our opinion the writer has confined himself too severely to the dry bones of the science, as he has described only the studies which actually lead to atomic positions, thereby eliminating many of the X-ray diffraction effects which are most interesting to chemists and technicians. Within its self-imposed limits, however, the book is excellent and is free from many of the faults which marred the former edition. In his previous description of X-ray methods, the writer did considerably less than justice to the ionisation spectrometer, refusing in fact to accept as reliable most of the analyses performed by means of that instrument. Events have forced him to revise his opinion, since (as he now remarks) "at present the ionisation spectrometer is the only source of quantitative data," and by its means many of the more complicated crystal structures—notably those of the silicates—have been completely elucidated. As a result of this change of heart, he now describes in detail many studies of organic crystals which, though published in 1924, received but the barest mention in his previous edition.

Another notable change is the dropping of his own space-group notation. Although this had several advantages over the original (Schoenflies) notation, it never won general acceptance, and it has become entirely superfluous now that by international agreement (the author concurring) Mauguin's notation is to be adopted.

Part II of the book, Results of Crystal Analysis, is of course almost completely new and is illustrated by many excellent diagrams. It has not been found possible within the limited size of the book to give a thorough discussion of the alloy systems, though the more definite intermetallic compounds have been described in their appropriate chapters. This omission, however, is of no great importance, since a lengthy description of all the work on alloys may be found in Ewald and Hermann's recently issued *Strukturberichte* (Leipzig, 1931). Some very useful tables give briefly the existing information concerning the crystals considered, together with a key to the appropriate references. The bibliography itself, contained in an appendix of some 78 pages, is classified according to the year of publication (alphabetically under the authors' names), and is as before a most valuable supplement to the text. There are a few omissions, including a paper by Manchot (*Z. f. anorg. Chem.*

124 1922 (333)) on silicon, one by Cohn and Tolksdorf (*Z. f. phys. Chem.* (B), **8** 1929 (331)) on ZrO_2, and a correction (*Z. f. Krist.* **70** 1929 (316)) by Mark and Mehner of their original paper on diathylphthalylketone (*Z. f. Krist.* **65** 1927 (461)). There is a useful subject index.

K. LONSDALE

Traité de Pyrométrie Optique. Par GUSTAVE RIBAUD. Pp. xvi + 488; 163 Figs. (Paris: Revue d'Optique théorétique et instrumentale, 1931.) Price 95 fr.

The book under review is vol. IV of the 5th Section of the *Encyclopédie Photométrique*. The scope is wider than the title suggests, since total radiation pyrometers are also described. The treatment of the subject is fairly complete in respect of Continental and American work on the subject, but the author ignores some of the British contributions.

An indication of the contents may be obtained from a consideration of the titles of some of the chapters. For example: The optical properties of bodies; Black body; Kirchhoff's law; Laws of radiation of a black body and the verification of the radiation laws; The measurement of the constant C_2; Emissive powers of non-black bodies; Colour temperature; Light emission from non-black bodies; Optical pyrometers and the system of extrapolation to high temperatures; Spectropyrometers; Effective wave-lengths; Choice of filter glasses and absorption glasses; The realisation of black body conditions in optical pyrometry; Lamps with strip filaments for the calibration of pyrometers; The palladium, platinum, and other fixed points; Total radiation pyrometers; The temperature measurement of some sources of illumination.

The author describes a number of pyrometers which are now of little more than historical interest. The general consensus of opinion at the present day amongst both scientific workers and industrial users is that the disappearing filament pyrometer is the most convenient and reliable instrument of the optical type. The author is therefore fully justified in devoting 50 pages to this instrument alone against 15 pages to three other types.

It is to be regretted that the author has not drawn a correct perspective view of the Wanner pyrometer (Fig. 104). The artist who drew this figure 30 years ago made an obvious mistake which should not be perpetuated indefinitely.

In spite of minor blemishes the book is one which will prove of value to those making a serious study of optical pyrometry, and one of the noteworthy features is the thorough treatment of the optical side of the subject.

E. G.

Text-Book on Spherical Astronomy. By W. M. SMART, M.A., D.Sc. Pp. xi + 414; 146 Figs. (Cambridge: University Press, 1931.) Price 21*s.* net.

Dr Smart's text-book bears the traditional title of *Spherical Astronomy*, but without any narrow restriction it contains those parts of astronomy which are adapted to elementary mathematical treatment and are therefore those which are most suitable for the serious student who is approaching the subject for the first time. It thus stands in the line of succession to Ball's *Spherical Astronomy* as contrasted with books of reference for professional astronomers. Ball's work had a characteristic originality which lent it charm for those who were capable of appreciating it, but this very quality made it less suitable for the beginner. Dr Smart, as a practical astronomer and an experienced teacher at Cambridge, has produced an eminently workmanlike and up-to-date text-book comprehensive

in scope and straightforward in treatment. If originality is not its most conspicuous feature, the book has compensating virtues which the average student should appreciate.

From a more critical point of view it may be felt that the work is lacking in unity of plan. In the last four chapters the reader jumps from astronomical photography to determination of position at sea, thence to binary star orbits, and finally to occultations and eclipses. This will illustrate the variety and extent of the topics treated. Within the limits of its own chapter each subject is well discussed, and if the several chapters are not always closely related in sequence the difficulty of covering so vast a science as astronomy must be remembered. For the same reason the author's choice of topics must be accepted though a personal view may incline to an alternative choice in a few instances. The diagrams are both copious and clear, and deserve special recognition.

H. C. P.

Lens Work for Amateurs (Fifth Edition). By HENRY ORFORD. Revised by A. LOCKETT. Pp. xv + 231; 231 Figs. (London: Sir Isaac Pitman & Sons, Ltd., 1931.) Price 3*s*. 6*d*. net.

The processes involved in the working of lenses and other optical elements are of considerable importance, and distinct advances have been made during and since the War. The literature dealing with the subject is scanty and spread over various periodicals, scientific societies' transactions, dictionaries, etc.

The present edition of Orford's book on *Lens Work for Amateurs* differs very little from the first edition published in 1894, and contains descriptions of methods and appliances (as on pp. 19, 23, 25, 52, 61) many of which are scarcely found in operation anywhere to-day. That the text, intended for amateurs, should include descriptions of modern factory methods is not expected, but one hoped to see the influence of such methods reflected in a book which occupies a position rendered somewhat prominent by the scarcity of literature on this subject.

The book has clearly been written by a practical man and contains useful information on the methods of making tools for, and the processes necessary for working, many optical systems such as eyepieces, microscope objectives, photographic lenses, etc., as well as prisms and flats. Useful information on the construction data of some of the systems is included and the text is freely illustrated by diagrams.

Occasionally the instructions could have been written in clearer English, and the somewhat popular descriptions of the principles and functions of various instruments are not so satisfactory as the more practical portions of the book.

The book is well printed and bound and a deal of information of considerable service to the amateur lens worker is to be found in it.

H. H. E